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Traveling-wave–standing-wave competition in a generalized complex Swift-Hohenberg equation

Víctor J. Sánchez-Morcillo, Germán J. de Valcárcel, and Eugenio Roldán
Departament d'Òptica, Universitat de València, Dr. Moliner 50, 46100 Burjassot, Spain

Kestutis Staliunas
Physikalisch Technische Bundesanstalt, Bundesalle 100, 38116 Braunschweig, Germany

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We study both analytically and numerically the competition between traveling-wave and standing-wave (SW) patterns in a unidimensional generalized complex Swift-Hohenberg equation (CSHE) appropriate for describing nondegenerate optical parametric oscillation. We find that SWs can win this competition because of the nonlinear resonance and nonlocal nonlinear phase modulation terms present in the CSHE. We also find a domain of bistability between the two types of solutions. A good agreement between analytical predictions and numerical simulation is found. [S1063-651X(98)50805-7]

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The transition from traveling waves (TWs) to standing waves (SWs) constitutes an interesting example of secondary symmetry breaking which has been the subject of several investigations in pattern formation studies [1–4]. At a first sight, TWs would be the preferred solution in infinitely extended phase unlocked systems while SWs would appear as a natural solution when the spatially extended system is either limited by hard boundary conditions or is phase locked.

This simple scenario can nevertheless be distorted by several factors. Appropriate hard boundary conditions (e.g., reflecting walls) can stabilize SWs to TWs [1], but intermediate states appear depending on the relative sizes of the defects induced by the boundary effects and of the transverse domain [2]; thus, the transition from TWs to SWs being nontrivial. Moreover, the symmetry breaking leading from TWs to SWs is possible in boundary-free phase unlocked systems as Riecke *et al.* have shown by temporally modulating a TW undergoing a Hopf bifurcation [3]. There is also experimental evidence for the opposite phenomenon, i.e., the unexpected transition from SWs to TWs in a rectangular container excited by a horizontal sinusoidal motion [4].

Here we show a boundary-free spontaneous symmetry breaking mechanism that destabilizes TWs to SWs. It is a bulk mechanism (based on the addition of some nonlinear resonance effects to the *complex* Swift-Hohenberg equation) at variance from previous reported mechanisms which, as far as we know, always rely on an external modulation of the system.

The complex Swift-Hohenberg equation (CSHE) [5] constitutes a model equation that is able of describing a large variety of pattern forming systems, phase unlocked spatially extended nonlinear optical systems (NLOS) among them (see a discussion about this in Ref. [6]). Outstanding examples of NLOSs describable by the CSHE are two-level lasers [7] and optical parametric oscillators (OPOs) [6,8,9]. Through this paper we concentrate on the generalized CSHE describing OPOs [6] in a slab waveguide configuration which confines the fields in one transverse direction, say y, diffraction acting along the x direction (light propagates along a direction perpendicular to the plane xy).

In optical parametric oscillation a $\chi^{(2)}$ medium placed inside an optical cavity transforms, through a down-conversion two-photon process, a coherent driving field of frequency ω_L into two fields of frequencies $f_1\omega_L$ (signal) and $f_2\omega_L$ (idler) $(f_1+f_2=1)$. When the fields are nearly resonant with three longitudinal modes of the cavity, OPOs are capable of forming complex spatial structures [6,8,10,11]. In Ref. [6] we have shown that the order parameter equation for an OPO is the following generalization of the CSHE:

$$\partial_t Y=(P-1)Y-|Y|^2Y-\tfrac{1}{2}(\Delta-\partial_x^2-\Delta_0|Y|^2)^2Y + \tfrac{1}{2}\Delta_0 Y(Y^*\partial_x^2 Y - Y\partial_x^2 Y^*), \qquad (1)$$

where Y is the order parameter (proportional to the signal field amplitude), P is a real parameter proportional to the plane-wave pump-field amplitude, Δ is an effective signal detuning, Δ_0 is the pump field–cavity detuning which acts as a nonlinear resonance parameter in Eq. (1), x is the transverse spatial coordinate, and t is the time (see Ref. [6] for details). All quantities appearing in Eq. (1) are adimensional. Equation (1) is valid for small, either positive or negative, values of the effective detuning Δ and close to the oscillation threshold $(P\approx 1)$. For the sake of simplicity, it has been assumed that the diffraction parameter for both the signal and idler fields are equal (the complete equation is given in Ref. [6]) but this assumption is not essential for the results described below.

Equation (1) differs from the standard CSHE by the terms multiplied by Δ_0: the nonlinear resonance brought about by the third term [6,10] and the nonlocal nonlinear phase modulation term [last term in Eq. (1)]. For null or small values of Δ_0 Eq. (1) reduces to the standard CSHE [6], where SWs are always unstable. In the following we show that the extra terms lead to the destabilization of TWs to SWs.

First we derive amplitude equations for two counterpropagating traveling waves. Assume pump values close to the oscillation threshold $P=1+\varepsilon^2 p$, and introduce a slow time scale $T=\varepsilon^2 t$. In order to take into account the different spatial scales associated with the fact that a band of modes have

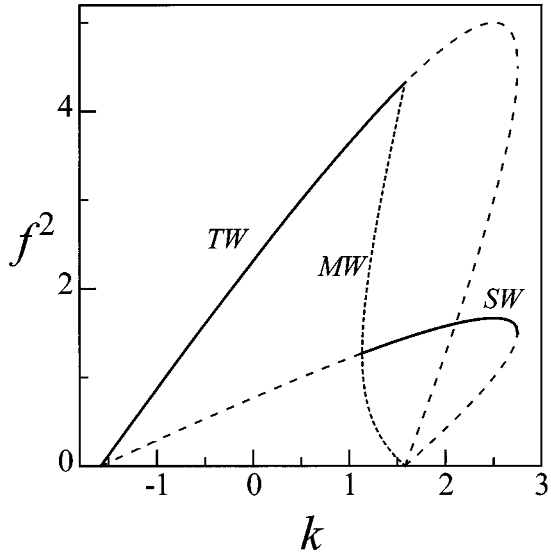

FIG. 1. Intensities of the TW, SW, and AW solutions as a function of the normalized wave number offset ($k=0$ corresponds to a linearly resonant wave) for $\mu=5$ as given by Eqs. (6a), (6b), and (7). The nonlinear resonance is clearly appreciated. Continuous (dashed) lines denote amplitude stable (unstable) solutions.

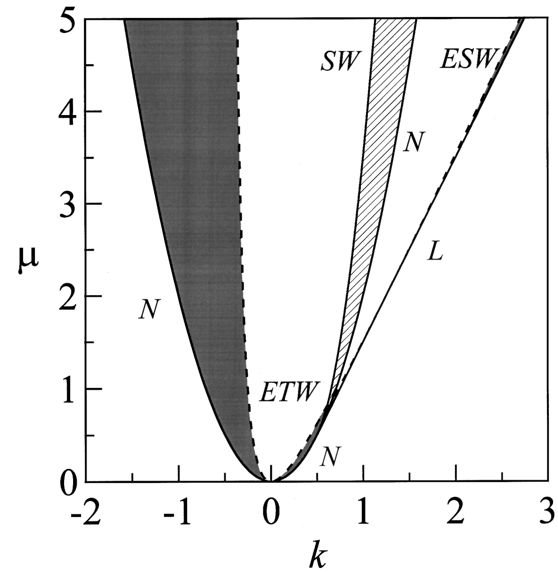

FIG. 2. Domains of existence and stability of the TW and SW solutions. The trivial solution loses stability at the boundary N (neutral stability curve). For $\mu<\frac{1}{2}$ TWs and SWs exist inside the region bounded by N. For $\mu>\frac{1}{2}$ TWs and SWs exist between the left branch of N and the curve L (due to the nonlinear resonance; see Fig. 1). TWs are amplitude stable in the region bounded by N. SWs are amplitude stable between the curve SW and L. Between curve SW and the right branch of N there is bistability between TWs and SWs. ETW and ESW denote the Eckhaus boundaries for TWs and SWs, respectively (the unstable regions are shadowed).

positive eigenvalues when $p>0$ it is necessary to introduce the multiple spatial scales ($x_0=x,\ u=\varepsilon x$). Thus the time and spatial derivatives transform as $\partial_t=\varepsilon^2\partial_T$, $\partial_x^2=\partial_{x_0}^2+\varepsilon(2\partial_u\partial_{x_0})$. It is further assumed that pump detuning is large enough in order to appreciate the influence of the nonlinear resonance, i.e., $\Delta_0=\varepsilon^{-1}\delta_0$.

Next we look for the equations governing the evolution of the slowly varying amplitudes of the pattern

$$Y=\varepsilon[F_+(u,T)e^{ik_0x_0}+F_-(u,T)e^{-ik_0x_0}]+O(\varepsilon^2), \quad (2)$$

where $k_0=\sqrt{-\Delta}$ is the wave number of the linearly resonant mode ($\Delta<0$ is assumed). Substitution of the previous scales and expansion into Eq. (1) leads to an infinite set of differential equations which gives the searched amplitude equations after imposing a solvability condition at the ε^3 order. The amplitude equations read

$$\begin{aligned}\partial_T F_\pm &= pF_\pm-(|F_\pm|^2+2|F_\mp|^2)F_\pm+2k_0^2\partial_u^2F_\pm\\ &\quad-\delta_0^2\left(\frac{1}{2}|F_\pm|^4+3|F_\pm|^2|F_\mp|^2+|F_\mp|^4\right)F_\pm-ik_0\delta_0\\ &\quad\times[-F_\pm\partial_uF_\pm^*+2F_\mp\partial_uF_\mp^*+\partial_u\\ &\quad\times(|F_\pm|^2+2|F_\mp|^2)]F_\pm . \end{aligned} \quad (3)$$

We concentrate on the solution $F_+=\tilde{f}_+e^{i\tilde{k}u}$, $F_-=\tilde{f}_-e^{-i\tilde{k}u}$ that corresponds to two counterpropagating waves with the same wave number. By introducing the rescaled variables and parameters

$$\tau=\delta_0^2T,\quad f_\pm\equiv\delta_0\tilde{f}_\pm,\quad k\equiv\delta_0k_0\tilde{k},\quad \mu\equiv\delta_0^2p, \quad (4)$$

the equations governing the evolutions of the amplitudes $f_\pm$ read

$$\begin{aligned}\partial_\tau f_\pm &= f_\pm[\mu-2k^2-(1-2k)(f_\pm^2+2f_\mp^2)\\ &\quad-(\tfrac{1}{2}f_\pm^4+3f_\pm^2f_\mp^2+f_\mp^4)], \end{aligned} \quad (5)$$

where $f_\pm$ have been taken real without loss of generality. Obviously, definitions (4) are valid provided that $\delta_0\neq0$.

Equations (5) admit two types of solutions apart from the trivial one ($f_+=0,\ f_-=0$): unidirectional solutions (TW), and bidirectional solutions [symmetric (SW) and asymmetric (AW)]. The unidirectional TW solutions are obtained by making either ($f_+=f_{\rm TW},\ f_-=0$) or ($f_+=0,\ f_-=f_{\rm TW}$) in Equations (5). The SW solution is obtained by making $f_+=f_-=f_{\rm SW}$ in Eqs. (5). These solutions read

$$f_{\rm TW}^2=(2k-1)\pm\sqrt{2\mu+1-4k}, \quad (6a)$$

$$f_{\rm SW}^2=\tfrac{1}{3}f_{\rm TW}^2. \quad (6b)$$

Finally, the asymmetric bidirectional solutions [($f_+=f_{\rm AW+},\ f_-=f_{\rm AW-}$) and its symmetric] read

$$f_{\rm AW\pm}^2=(2k-1)\pm\sqrt{1-\tfrac{2}{3}\mu-4k+\tfrac{16}{3}k^2}. \quad (7)$$

Both the TW (6a) and the SW (6b) solutions can be bivalued depending on the pump value μ. If $\mu<\frac{1}{2}$ both solutions are unique and exist for

$$k\in[-k_N,k_N],\quad k_N=\sqrt{\tfrac{1}{2}\mu}. \quad (8)$$

For $\mu>\frac{1}{2}$ both TW and SW exist for $k\in[-k_N,k_L]$ being bivalued for

$$k \in [k_N, k_L], \quad k_L = \tfrac{1}{4}(2\mu + 1). \tag{9}$$

These results are summarized in Figs. 1 and 2. Note that the bidirectional asymmetric solution (7) connects the TW and SW solutions.

Next we summarize the result of the linear stability analysis of the previous solutions. The trivial solution ($f_+ = 0$, $f_- = 0$) loses stability at $\mu = 0$, and for $\mu > 0$ it is unstable for $k \in [-k_N, k_N]$. The asymmetric bidirectional solution AW (7) is always unstable, as well as the lower branches of the TW and SW solutions [those corresponding to the minus sign in Eqs. (6a) and (6b)]. The single source of instability of the upper branch of the TW solution (6a) is the growth of the counterpropagating wave. The TW solution is stable in the domain where it is single valued (8) and unstable where it is double valued (9). Thus for $\mu < \frac{1}{2}$ the TW is always stable to the SW, and for $\mu > \frac{1}{2}$ it is unstable where it is bivalued. Regarding the SW solution (6b) it is stable only for $\mu > \frac{1}{2}$ within the domain given by

$$k \in [k_{\mathrm{SW}}, k_L], \quad k_{\mathrm{SW}} = \tfrac{1}{8}(3 + \sqrt{8\mu - 3}). \tag{10}$$

Hence there exists a bistability domain between the TW and SW patterns for $k \in [k_{\mathrm{SW}}, k_N]$ (see Fig. 2). The fact that the (unstable) AW branch connects the instability points corresponding to both the TW and SW solutions indicates that these points correspond to subcritical pitchfork bifurcations.

There remains to assess the existence of phase instabilities. As we are considering only one transverse dimension, the only possible phase instability is the Eckhaus one. A straightforward calculation leads to the conclusion that TWs and SWs are Eckhaus unstable for

$$\mathrm{TW}: \mu < 2k^2 \frac{4k+3}{(2k+1)^2}, \tag{11}$$

$$\mathrm{SW}: \mu < \tfrac{1}{16}[4k^2 + 76k + 21 \pm (2k+7)\sqrt{4k^2 + 60k + 9}]. \tag{12}$$

These boundaries are depicted in Fig. 2 (dashed lines) where the Eckhaus unstable regions are shadowed. Note that this instability does not affect the TW-SW transition, but for $\mu \gtrsim \frac{1}{2}$.

In order to check the validity of these analytical results, we have carried out numerical integrations of the OPO microscopic equations [6] (see [6] for the integration procedure) for parameter values compatible with the domain of validity of the CSHE (1). Figure 3 shows the intensity f^2 of the calculated TW and the SW patterns (see caption for parameter values). Filled symbols correspond to stable solutions. A good agreement is apparent when comparing with Fig. 1. There is a relatively small quantitative discrepancy between the theoretical and numerical values for the Eckhaus boundary (numerically the Eckhaus unstable domain for SWs is broader). This discrepancy can be attributed to the fact that we are integrating the OPO microscopic equations and not the generalized CSHE.

We can finally ask which is the role played by each of the two extra terms added to the CSHE on the stabilization of SWs and on the bistability between SWs and TWs. For that, we have investigated the competition between TWs and SWs in the case in which the nonlocal nonlinear phase modulation term is absent, i.e., we have studied the modified CSHE

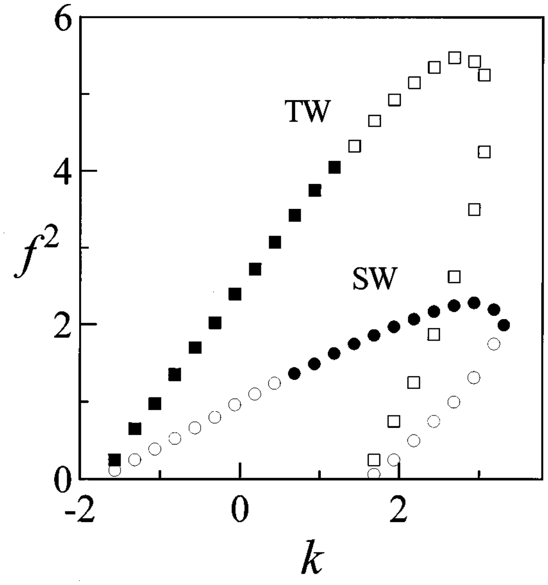

FIG. 3. Intensities of the TW and SW solutions corresponding to numerical simulations performed with the OPO microscopic equations for $\Delta_0 = 5$ and $P = 1.2$, for which $\mu = 5$ [see (4)], as in Fig. 1. For computational convenience the normalized wave number offset k was varied by varying the effective detuning Δ (or, equivalently, by varying the resonant wave number k_0) while keeping the total wave number $(k_0 + \tilde{k})$ of the roll fixed (see [6] for details). Closed (open) symbols denote amplitude stable (unstable) solutions.

$$\partial_t Y = (P-1)Y - |Y|^2 Y - \tfrac{1}{2}(\Delta - \partial_x^2 - \Delta_0 |Y|^2)^2 Y. \tag{13}$$

In this case the amplitude equations for the two counterpropagating waves are similar to Eq. (5) but with the last term replaced by $-\frac{1}{2}(f_\pm^4 + 5f_\pm^2 f_\mp^2 + 3f_+^4)$. In this case TWs are always stable to SWs while SWs are stable only for $\mu > \frac{1}{2}$ in the domain given by

$$k \in [k'_{\mathrm{SW}}, k_L], \quad k'_{\mathrm{SW}} = -\tfrac{3}{2} + \sqrt{2\mu + 3}. \tag{14}$$

Thus, although TWs and SWs stably coexist (for $k \in [k'_{\mathrm{SW}}, k_N]$), there is no transition TW→SW. Hence, the nonlinear resonance is at the origin of the stabilization of SWs, and the loss of stability of TWs must be attributed to the joint action of the nonlinear resonance and the nonlocal nonlinear phase modulation.

In this paper we have demonstrated a bulk mechanism for the symmetry breaking that leads from traveling waves to standing waves in a one-dimensional complex Swift-Hohenberg equation, which describes, e.g., optical parametric oscillators in a slab waveguide geometry. The mechanism consists of the joint action of a nonlinear resonance (which is responsible for the stabilization of the SWs) and a nonlocal nonlinear phase modulation (which is necessary for the destabilization of TWs). Thus, the symmetry breaking is not due to any external action on the system (as hard boundary conditions [2] or external modulation [3]). Our analytical predictions are in good agreement with numerical integrations of the microscopic equations of optical parametric oscillators.

Finally, let us comment that when two spatial dimensions are allowed, preliminary numerical and analytical studies show that SWs are always unstable, mostly due to a zigzag instability. This instability, however, manifests only in the transient periods, since its actual role is in fact to mediate the disappearance of the SW as an attractor of the system, resting the TW (or some of its long wavelength perturbed states) the global attractor. This negative effect can be compensated, however, by the inclusion of diffraction terms in Eq. (1), which in the OPO case appear as far as the diffraction coefficients of signal and idler fields are unequal [6].

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